Entropy reduction from a detailed fluctuation theorem for a nonequilibrium stochastic system driven under feedback control

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We show explicitly the entropy reduction from a detailed fluctuation theorem for the general stochastic system driven by nonequilibrium process under feedback control. The effect of interaction of the feedback controller with the system is to reduce the entropy of the system. We define the entropy reduction for the single trajectory and show that the overall entropy production for the stochastic system with feedback control can be either positive or negative. The negative entropy production has been studied earlier for a simple system with velocity dependent feedback control [K. H. Kim and H. Qian, Phys. Rev. Lett. **93**, 120602 (2004)]. Our general approach provides the overall positive or negative entropy production irrespective of velocity dependent and position dependent feedback control.

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I. INTRODUCTION

The evolution of the physical systems can be modified or controlled by repeated operation of an external agent called controller [1, 2]. The action of the controller is to regulate the system dynamics and increase its performance. The controller can operate on the system blindly or it can use information about the state of the system. The former is known as the open loop controller and the latter one is called as feedback or closed loop controller. The feedback controller measures the partial performance of the system, and its action on the system depends on the outcome of measurements [3]. Recent advances in nanotechnology allows the active control of the position and velocity of the single molecule by a feedback system [4, 5]. The proper utilization of the information about the state of the system in feedback control effectively improves the system performance [1–3].

In last two decades, the concept of fluctuation theorems has become an active research area in statistical physics for advanced theoretical understanding and experimental verifications [6, 7]. This theorem generally establishes the connections between the nonequilibrium stochastic fluctuations of system and its dissipative properties. Further it establishes rigorous identities for the nonequilibrium averages of thermodynamic observables such as work, heat, entropy, or current [8–10].

Consider a system initially in equilibrium at temperature (inverse) $\beta = 1/k_BT$ (k_B is the Boltzmann constant) which is externally driven from its initial state to final state by nonequilibrium process. Let $P[\Gamma(t)]$ be the probability of the phase space trajectory, $\Gamma(t)$ for the system driven between the two states in forward direction. This satisfies the detailed fluctuation theorem [9–12],

$$\frac{P[\Gamma(t)]}{P^{\dagger}[\Gamma^{\dagger}(t)]} \; = \; e^{\sigma[\Gamma(t)]}, \tag{1}$$

where $\sigma[\Gamma(t)]$ is the positive entropy production of the driven system and $P^{\dagger}[\Gamma^{\dagger}(t)]$ is the probability of the phase space trajectory $\Gamma^{\dagger}(t)$, for the system driven in the reversed direction. This is the direct relation between the entropy production and the ratio of probabilities for the forward and the reversed trajectories [10, 11].

The presence of feedback control in a physical system in general modifies both the fluctuation theorem and the nonequilibrium averages such as Jarzynski equality [5, 12]. We have shown recently that the feedback control in a physical system should preserve the detailed fluctuation theorem if the system has the same feedback information measure in both directions [13]. Our results are based on the assumption that the system with feedback control should locally satisfy the detailed fluctuation theorem. Such an assumption leads to a positive entropy production from a detailed fluctuation theorem for the system with feedback control. However, the effect of the interaction of the controller with the system is to reduce the entropy of the system [3, 5] which was not captured in our earlier results [13].

The entropy production can be positive or negative depending on the feedback control. The negative entropy production can happen generally in a system with velocity dependent feedback control (VFC) such as molecular refrigerator [5]. There is a general belief that the entropy reduction in velocity dependent feedback control (VFC) is significantly different from the stochastic systems with position dependent feedback control (PFC) [5]. Since the entropy reduction in the system due to the information used by the feedback controller is quite general for both VFC and PFC, it will be worth to discuss the entropy reduction from a fluctuation theorem due to feedback control in a general framework.

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In following a few works on the single trajectory entropy for the stochastic system [9, 10, 14], in this paper, we define for the first time the entropy reduction of the single trajectory, and show explicitly the entropy reduction from a detailed fluctuation theorem for the general stochastic system driven under feedback control. The explicit appearance of the entropy reduction term in fluctuation theorem is an important result of this work. In addition, our approach allows one to compute the overall positive or negative entropy production irrespective of PFC or VFC.

II. SINGLE TRAJECTORY ENTROPY REDUCTION DUE TO FEEDBACK CONTROL

The feedback control enhances our controllability of small thermodynamic systems [1–3]. Whenever the controller measures the partial state of the system, the result of the measurement determines the action of the controller. The additional information on the system provided by the measurement further determines the state of the system. Suppose, the controller performs a measurement on a stochastic thermodynamic system at time t_m. Let Γ_m be the phase-space point of the system at that time, $P[\Gamma_m]$ its probability, and y the measurement. Depending on the controller measurements, the outcome y can occur with a probability $P[y]$ [12]. The information obtained by the controller can be characterized by the mutual (feedback) information measure [12, 13],

$$I[y, \Gamma_m] = ln \left[\frac{P[y|\Gamma_m]}{P[y]}\right], \tag{2}$$

where $P[y|\Gamma_m]$ is the conditional probability of obtaining outcome y on condition that the state of the system is Γ_m.

In experiments and simulations, the path connecting the two states of the system in the time period τ can be obtained by pulling the system from one state to another along a switching path. This can be parameterized using the switching protocol path variable λ [8, 12]. The switching rate determines whether the switching process is an equilibrium (infinitely slow) or nonequilibrium (fast) process. If the experiments are performed under feedback control, the switching control parameter λ depends on the outcome y after t_m [12]. That is, whenever the controller makes measurements, there is a corresponding change in the switching parameter for the next time step, which is denoted as $\lambda_{(t;y)}$. After each measurement, the value of outcome y should be fixed and the corresponding switching parameter $\lambda_{(t;y)}$ does not change until the controller make the another measurement.

Let $P_{\lambda_{(t;y)}}[\Gamma(t)]$ be the probability of the phase space trajectory, $\Gamma(t)$ in forward direction for the switching protocol $\lambda_{(t;y)}$. Since the entropy of the system is reduced by feedback control [3], we define the single trajectory entropy reduction for the protocol $\lambda_{(t;y)}$ as

$$\sigma_r[\Gamma(t)] \;=\; ln \left[\frac{P_{\lambda^\star}[\Gamma(t)]}{P_{\lambda_{(t;y)}}[\Gamma(t)]}\right]. \tag{3}$$

Here $P_{\lambda^\star}[\Gamma(t)]$ is the probability of the phase space trajectory for the system without feedback control driven by an arbitrary chosen switching protocol $\lambda^\star$ (see, Eq.(1)). It should be noted that prior to the controller measurement the feedback experiment initially started with the same protocol $\lambda^\star$. Our definition of entropy reduction has a definite meaning in the sense that if the system has no effect of feedback control, $P_{\lambda_{(t;y)}}[\Gamma(t)] = P_{\lambda^\star}[\Gamma(t)]$, which results in zero entropy reduction. However, one can see latter from Eq.(10) that there should be a single trajectory entropy production. If the driven system strongly influenced by the feedback controller, $P_{\lambda_{(t;y)}}[\Gamma(t)]$ should be significantly different from $P_{\lambda^\star}[\Gamma(t)]$, which eventually results in reduction in entropy [3].

Starting from the same switching protocol $\lambda^\star$ one can perform the feedback control experiment in reverse direction by driving the system from the final equilibrium state of the forward switching process to its initial equilibrium state. Let $P^\dagger_{\lambda^\star}[\Gamma^\dagger(t)]$ be the probability of the phase space trajectory, $\Gamma^\dagger(t)$ of the system without feedback control driven in the reverse direction for the switching protocol $\lambda^\star$ (see, Eq.(1)). For every trajectory in the forward direction there should be a corresponding trajectory in the reverse direction [9–12]. If we perform the feedback controller measurements for a given system in the reverse direction, the single trajectory entropy reduction in the reverse direction is

$$\sigma^\dagger_r[\Gamma^\dagger(t)] \;=\; -\sigma_r[\Gamma(t)], \tag{4}$$

From the definition of Eq.(3), the single trajectory entropy reduction in reverse direction is given by

$$\sigma^\dagger_r[\Gamma^\dagger(t)] \;=\; ln \left[\frac{P^\dagger_{\lambda^\star}[\Gamma^\dagger(t)]}{P^\dagger_{\lambda^\dagger_{(t;y^\dagger)}}[\Gamma^\dagger(t)]}\right], \tag{5}$$

where $P^\dagger_{\lambda^\dagger_{(t;y^\dagger)}}[\Gamma^\dagger(t)]$ is the probability of the phase space trajectory in reverse direction for the switching protocol $\lambda^\dagger_{(t;y^\dagger)}$. The measurement outcome $y^\dagger$ can occur with probability $P^\dagger[y^\dagger]$. The mutual information measure due to feedback control in reverse direction is given by

$$I^\dagger[y^\dagger, \Gamma^\dagger_m] = ln \left[\frac{P^\dagger[y^\dagger|\Gamma^\dagger_m]}{P^\dagger[y^\dagger]}\right], \tag{6}$$

where $P^\dagger[y^\dagger|\Gamma^\dagger_m]$ is the conditional probability of obtaining outcome $y^\dagger$ on the condition that the state of the system is $\Gamma^\dagger_m$.

Using Eq.(1), Eq.(5) can be rewritten as

$$\begin{aligned}\sigma_r^\dagger[\Gamma^\dagger(t)] &= ln\left[\frac{P^\dagger_{\lambda^\star}[\Gamma^\dagger(t)]}{P_{\lambda^\star}[\Gamma(t)]}\right] \\ &+ ln\left[\frac{P_{\lambda^\star}[\Gamma(t)]}{P^\dagger_{\lambda^\dagger_{(t;y^\dagger)}}[\Gamma^\dagger(t)]}\right] \\ &= -\sigma[\Gamma(t)] + ln\left[\frac{P_{\lambda^\star}[\Gamma(t)]}{P^\dagger_{\lambda^\dagger_{(t;y^\dagger)}}[\Gamma^\dagger(t)]}\right].\end{aligned}$$

Therefore,

$$ln\left[\frac{P_{\lambda^\star}[\Gamma(t)]}{P^\dagger_{\lambda^\dagger_{(t;y^\dagger)}}[\Gamma^\dagger(t)]}\right] = \sigma_r^\dagger[\Gamma^\dagger(t)] + \sigma[\Gamma(t)]. \tag{7}$$

The left hand side of the above equation can be rewritten as

$$ln\left[\frac{P_{\lambda^\star}[\Gamma(t)]}{P_{\lambda_{(t;y)}}[\Gamma(t)]}\right] + ln\left[\frac{P_{\lambda_{(t;y)}}[\Gamma(t)]}{P^\dagger_{\lambda^\dagger_{(t;y^\dagger)}}[\Gamma^\dagger(t)]}\right] = \sigma_r^\dagger[\Gamma^\dagger(t)] + \sigma[\Gamma(t)]. \tag{8}$$

Using Eq.(3) and Eq.(4), the above equation becomes,

$$\frac{P_{\lambda_{(t;y)}}[\Gamma(t)]}{P^\dagger_{\lambda^\dagger_{(t;y^\dagger)}}[\Gamma^\dagger(t)]} = e^{S_r[\Gamma(t)]}, \tag{9}$$

where

$$S_r[\Gamma(t)] = -2\sigma_r[\Gamma(t)] + \sigma[\Gamma(t)] \tag{10}$$

whose reversed part

$$S_r^\dagger[\Gamma^\dagger(t)] = -S_r[\Gamma(t)]. \tag{11}$$

It is clear from Eq.(10) that if the feedback control has no effect on the system, $\sigma_r[\Gamma(t)] = 0$, however, there should be a single trajectory entropy production $S_r[\Gamma(t)] = \sigma[\Gamma(t)]$.

Under identical experimental conditions in the forward and reverse directions, we assume that the error in the controller measurements outcomes y and $y^\dagger$ should be the same. In such a case,

$$P[y] = P^\dagger[y^\dagger]. \tag{12}$$

Combining Eq.(2), Eq.(6) and Eq.(12), we can obtain

$$\frac{P[y|\Gamma_m]}{P^\dagger[y^\dagger|\Gamma_m^\dagger]} = e^{I[y,\Gamma_m]-I^\dagger[y^\dagger,\Gamma_m^\dagger]}. \tag{13}$$

For the controller measurement condition $I[y,\Gamma_m] = I^\dagger[y^\dagger,\Gamma_m^\dagger]$ [13], the above equation becomes,

$$P[y|\Gamma_m] = P^\dagger[y^\dagger|\Gamma_m^\dagger]. \tag{14}$$

If we obtained the same measurement outcome y' [12, 13] of phase point Γ' in the forward direction and $\Gamma'^\dagger$ in the reverse direction then Eq.(9) and Eq.(14) become,

$$\frac{P_{\lambda_{(t;y')}}[\Gamma(t)]}{P^\dagger_{\lambda^\dagger_{(t;y')}}[\Gamma^\dagger(t)]} = e^{S_r[\Gamma(t)]} \tag{15}$$

$$P[y'|\Gamma'] = P^\dagger[y'|\Gamma'^\dagger]. \tag{16}$$

In what follows we use the above equations and obtain the overall entropy production from a detailed fluctuation theorem due to feedback control in both the directions.

III. DETAILED FLUCTUATION THEOREM UNDER FEEDBACK CONTROL

Let $P[\widetilde{S}_r]$ be the probability of obtaining $S_r[\widetilde{\Gamma}]$ from the repeated feedback control experiment in the forward direction. This probability can be obtained from the joint distribution of $\Gamma(t)$ and y is given by [12, 13]

$$\begin{aligned}P[\widetilde{S}_r] &= \int P[y'|\Gamma']P_{\lambda_{(t;y')}}[\Gamma(t)] \\ &\quad \delta(S_r[\Gamma(t)] - S_r[\widetilde{\Gamma}])\, dy'\ D[\Gamma(t)],\end{aligned} \tag{17}$$

where $\delta(x)$ is the Dirac delta function which has a property $\delta(-x) = \delta(x)$. Using Eq.(15), the above equation becomes,

$$\begin{aligned}P[\widetilde{S}_r] &= \int e^{S_r[\Gamma(t)]}P[y'|\Gamma']P^\dagger_{\lambda^\dagger_{(t;y')}}[\Gamma^\dagger(t)] \\ &\quad \delta(S_r[\Gamma(t)] - S_r[\widetilde{\Gamma}])dy'\ D[\Gamma(t)],\end{aligned}$$

$$\begin{aligned}P[\widetilde{S}_r] &= e^{S_r[\widetilde{\Gamma}]}\int P[y'|\Gamma']P^\dagger_{\lambda^\dagger_{(t;y')}}[\Gamma^\dagger(t)] \\ &\quad \delta(S_r[\Gamma(t)] - S_r[\widetilde{\Gamma}])dy'\ D[\Gamma(t)],\end{aligned}$$

Using Eq.(11) and Eq.(16), the above integral can be rewritten as,

$$\begin{aligned}P[\widetilde{S}_r] &= e^{S_r[\widetilde{\Gamma}]}\int P^\dagger[y'|\Gamma'^\dagger]P^\dagger_{\lambda^\dagger_{(t;y')}}[\Gamma^\dagger(t)] \\ &\quad \delta(S_r^\dagger[\Gamma^\dagger(t)] - S_r^\dagger[\widetilde{\Gamma}^\dagger]) \\ &\quad dy'\ D[\Gamma^\dagger(t)].\end{aligned} \tag{18}$$

Since $\delta(-x) = \delta(x)$ and $D[\Gamma^\dagger(t)] = D[\Gamma(t)]$ [12], we can obtain the overall entropy production from a detailed

fluctuation theorem under feedback control in both directions as

$$\frac{P[\widetilde{S}_r]}{P^{\dagger}[-\widetilde{S}_r]} \;=\; e^{S_r[\widetilde{\Gamma}]}, \tag{19}$$

where $P^{\dagger}[-\widetilde{S}_r]$ is the probability of obtaining $S_r^{\dagger}[\widetilde{\Gamma}^{\dagger}]$ from the repeated feedback control experiment in the reverse direction. This result explicitly provides the entropy reduction in the controlled system due to the external agent that operates on it. We can rewrite Eq.(19) simply as

$$\frac{P[-2\sigma_r+\sigma]}{P^{\dagger}[-(-2\sigma_r+\sigma)]} \;=\; e^{-2\sigma_r+\sigma}. \tag{20}$$

From the above relation, we can discuss the different entropy reduction conditions as follows. If $\sigma_r < \sigma/2$, S_r is always positive, which provides the overall positive entropy production. It is clear that the positive entropy production as obtained in our earlier work [13] for the system with feedback control in both directions is a special case of the present derivation. If $\sigma_r = \sigma/2$, the overall entropy production is zero. This may be a condition for equilibrium [5]. If $\sigma_r > \sigma/2$, S_r is always negative, which provides the overall negative entropy production. This situation can happen quite generally in velocity dependent feedback control experiment (molecular refrigerator) [5]. This shows that our general result is valid for both the velocity dependent and position dependent feedback control. The overall entropy production can be positive or negative depending upon the interaction of the system with the feedback controller.

Finally, using Eq.(20), it is straightforward to obtain the integral fluctuation theorem [10],

$$\langle e^{-(-2\sigma_r+\sigma)}\rangle = 1. \tag{21}$$

If the feedback control has no effect on the system, $\sigma_r = 0$, then we get positive entropy production detailed fluctuation theorem [15]

$$\frac{P[\sigma]}{P^{\dagger}[-\sigma]} = e^{\sigma}. \tag{22}$$

IV. CONCLUSION

We have defined the single trajectory entropy reduction for the nonequilibrium stochastic system driven under feedback control. Our result (Eq.20) explicitly shows the entropy reduction from a detailed fluctuation theorem for a stochastic system driven under feedback control. Our general result is valid for both the velocity dependent and the position dependent feedback control. This allows one to compute the overall entropy production which can be either positive or negative.

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both the directions, one can obtain from Eq.(13) that

$$I[y, \Gamma_m] = ln \left[\frac{P[y|\Gamma_m]}{P^\dagger[y^\dagger|\Gamma_m^\dagger]} \right].$$

This relation is similar to the change of uncertainty as defined in Ref.[16] (see, cf Eq.(15) of Ref.[16] for single loop feedback).

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